REPORT 1149

ON TRANSONIC FLOW PAST A WAVE-SHAPED WALL[1]

By Carl Kaplan

SUMMARY

The present report is an extension of a previous investigation (described in NACA Rep. 1069) concerned with the solution of the nonlinear differential equation for transonic flow past a wavy wall. In the present work several new notions are introduced which permit the solution of the recursion formulas arising from the method of integration in series. In addition, a novel numerical test of convergence, applied to the power series (in the transonic similarity parameter) representing the local Mach number distribution at the boundary, indicates that smooth symmetrical potential flow past the wavy wall is no longer possible once the critical value of the stream Mach number has been exceeded.

INTRODUCTION

In NACA Report 1069 (ref. 1) the writer considered the problem of two-dimensional transonic flow past an infinitely long sinusoidal solid boundary. The problem was treated in the physical plane and the purpose was to investigate whether or not the flow past the wavy wall remains a smooth symmetrical type of potential flow when the undisturbed-stream Mach number exceeds its critical value. By a smooth type of potential flow is meant one for which the velocity potential, say, and its first derivatives are single-valued and continuous; that is, there are no discontinuities of the nature of shock waves.

Initially, the Prandtl-Busemann small-perturbation method was applied and the velocity potential developed inclusive of the third power in the "thickness" coefficient $\epsilon=\frac{a}{\lambda/2\pi}$ where

a amplitude of wavy wall

λ wave length of wavy wall

The velocity potential was then referred to the critical speed of sound c_{cr}, the coefficient ϵ was replaced by $\frac{k(1-M^2)^{3/2}}{\gamma+1}$ where $k=\frac{(\gamma+1)\epsilon}{(1-M^2)^{3/2}}$ is the transonic similarity parameter, and terms involving powers of $1-M_\infty^2$ higher than the first were neglected (main assumption for transonic flow). This simplified or transonic form of the Prandtl-Busemann solution was shown to be identical with the one obtained by solving the simplified nonlinear differential equation for transonic flow, with the boundary condition taken not at the wave-shaped wall but at the flat plate corresponding to vanishing amplitude. The calculation was carried through the sixth power in the transonic similarity parameter k and corresponds to the insuperable task of obtaining the Prandtl-Busemann solution to the sixth power in the thickness coefficient ϵ. Thus each iteration step of the Prandtl-Busemann method contributes to the transonic form of the solution, which may therefore be considered a result of thin-profile theory with disturbances not necessarily small compared with $1-M_\infty^2$. The main conclusion reached in reference 1 was that the transonic similarity parameter k must be less than $\frac{4}{3}$—a value still somewhat greater than the critical value 0.83770 calculated there.

The purpose of the present work is to express the solution of the problem of transonic flow past the wavy wall in a form more suitable for general considerations and to prove that the assumed smooth symmetrical type of potential flow cannot exist at stream Mach numbers beyond the critical value.

SYMBOLS

x, y	nondimensional rectangular Cartesian coordinates
a	amplitude of wavy wall
λ	wave length of wavy wall
$\lambda/2\pi$	reference element of length
ϵ	"thickness" coefficient of wavy wall, $\frac{a}{\lambda/2\pi}$
γ	ratio of specific heats at constant pressure and constant volume
U	stream velocity
M	local Mach number
M_∞	undisturbed-stream Mach number
k	transonic similarity parameter, $\frac{(\gamma+1)\epsilon}{(1-M^2)^{3/2}}$
c_{cr}	critical speed of sound
ϕ	velocity potential of flow
$f(x,y)$	transonic disturbance potential
$f_n(y)$	functions of y only, related to $f(x,y)$
$A_{q\ r}^{n\ p}$	numerical coefficients
$A_{q}^{n\ p}$	generating functions of k, $\sum_{r=0}^{\infty} A_{q\ p+r}^{n\ p} k^{2r}$
$A_{m\ p}$	functions of dummy variable r, $\sum_{n=1,m}^{\infty} A_{n\ -m}^{n\ p} r^n$, lower label starts from 1 when m is negative and from m when m is positive

Primes denote differentiation with respect to independent variable.

[1] Supersedes NACA TN 2748, "On Transonic Flow Past a Wave-Shaped Wall" by Carl Kaplan, 1952.

ANALYSIS

GENERAL FORMULAS

If the undisturbed stream is in the direction of the positive x-axis, then the velocity potential ϕ referred to the critical velocity c_{cr} can be written as (see ref. 1)

$$\phi=x+\frac{1}{\gamma+1}(1-M_\infty^2)f(x,y)$$

where the second term on the right-hand side is a disturbance velocity potential and implies that terms involving powers of $1-M_\infty^2$ higher than the first have been neglected. The differential equation for transonic flow satisfied by the function $f(x,y)$ is obtained from the general differential equation for compressible flow and takes the following simplified nonlinear form:

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x^2} \tag{1}$$

The boundary conditions to be fulfilled by $f(x,y)$ are as follows:

$$\left.\begin{array}{ll}\left.\begin{array}{l}\dfrac{\partial f}{\partial x}=-1\\[2ex] \dfrac{\partial f}{\partial y}=0\end{array}\right\} & (\text{at } y=\infty)\\[4ex] \dfrac{\partial f}{\partial y}=-k\sin x & (\text{at } y=0,\ -\infty<x<\infty)\end{array}\right\} \tag{2}$$

Here x and y are nondimensional rectangular Cartesian coordinates simply related to the physical plane coordinates X and Y by means of the transformation

$$x=X$$

$$y=(1-M_\infty^2)^{1/2}Y$$

and the equation of the wavy wall is $Y=\epsilon\cos X$ or $y=(1-M_\infty^2)^{1/2}\epsilon\cos x$. Clearly, $f(x,y)$ involves only the variables x and y and the transonic similarity parameter k.

The most general form for the function $f(x,y)$ to ensure symmetrical potential flow past the wavy wall is the following (see fig. 1):

$$f(x,y)=-x+\sum_{n=1}^{\infty}f_n\sin nx \tag{3}$$

where the f_n are functions of y only. When this form for $f(x,y)$ is substituted into equation (1) and the coefficients of the separate harmonic terms $\sin nx$ are equated to zero,

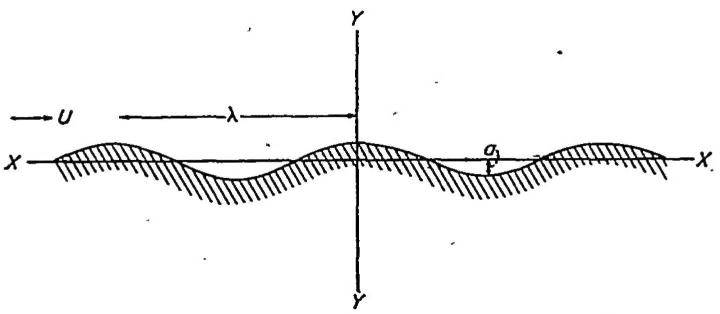

FIGURE 1.—Wave-shaped wall.

the following system of second-order nonlinear ordinary differential equations for f_n results:

$$f_n''-n^2f_n=-\frac{1}{4}n\sum_{m=1}^{n-1}m(n-m)f_mf_{n-m}-\frac{1}{2}n\sum_{m=1}^{\infty}m(n+m)f_mf_{n+m}\qquad(n=1,2,\ldots\infty) \tag{4}$$

Before proceeding to the solution of these equations, several formulas of general interest and subsequent use are given. They have been derived in reference 1. The local Mach number distribution is given by

$$1-M^2=-(1-M_\infty^2)\frac{\partial f}{\partial x} \tag{5}$$

The equation from which the critical value of the transonic similarity parameter is calculated follows from equation (5) by taking $x=0$, $y=0$, and $M=1$; that is,

$$\left(\frac{\partial f}{\partial x}\right)_{\substack{x=0\\y=0}}=0$$

or with the aid of equation (3),

$$\sum_{n=1}^{\infty}nf_n(0)=1 \tag{6}$$

The pressure coefficient

$$C_{p,M_\infty}=\frac{p-p_\infty}{\frac{1}{2}\rho_\infty U^2}$$

is given by

$$C_{p,M_\infty}=-\frac{2}{\gamma+1}(1-M_\infty^2)\left(1+\frac{\partial f}{\partial x}\right) \tag{7}$$

INTEGRATION IN SERIES FOR THE FUNCTIONS f_n

In reference 1, equations (4) were solved by an iteration procedure and approximate expressions for f_1 to f_6 were obtained. An examination of these expressions showed that the general form of f_n is

$$f_n=\sum_{p=0}^{\infty}e^{-(2p+n)y}\sum_{q=0}^{\substack{n-1\\2p+n-2}}y^q\sum_{r=p}^{\infty}A_{q\,r}^{n\,p}\,k^{n+2r}\qquad(n=1,2,\ldots\infty) \tag{8}$$

where, if $p=0$, the upper limit of q is $n-1$ and, if $p\neq0$, the upper limit of q is $2p+n-2$. The four-labeled coefficients $A_{q\,r}^{n\,p}$ are real numbers calculated from recursion formulas obtained from the system of differential equations (4) and the boundary condition at the surface of the wavy wall. The boundary condition at $y=\infty$ is automatically satisfied by the form of f_n; whereas the boundary condition at the wall takes the form

$$(f_n')_{y=0}=\begin{matrix}-k\\0\end{matrix}\quad\left.\begin{pmatrix}n=1\\n\neq1\end{pmatrix}\right\} \tag{9}$$

Inserting the expression for f_n given by equation (8) into equations (9) yields immediately the following results:

$$\left.\begin{array}{l}A_{0\,0}^{1\,0}=1\\[1ex] \displaystyle\sum_{p=0}^{r}(2p+n)A_{0\,r}^{n\,p}=\sum_{p=0,1}^{r}A_{1\,r}^{n\,p}\qquad(n=1,2,\ldots\infty)\end{array}\right\} \tag{10}$$

where, if $n=1$, the lower limit of p on the right-hand side is unity and, if $n\neq 1$, the lower limit of p is zero. Also, if $n=1$, the upper limit r of p goes from 1 to ∞ and, if $n\neq 1$, r goes from 0 to ∞.

By elementary manipulation of series, the second of equations (10) can be replaced by the following more useful forms:

$$\left.\begin{aligned} &nA^{n\;0}_{0\;0}=A^{n\;0}_{1\;0} && (n=2,\,3,\,\ldots\,\infty)\\ &nA^{n\;0}_{0\;r}=\sum_{p=0,1}^{r}A^{n\;p}_{1\;r}-\sum_{p=1}^{r}(2p+n)A^{n\;p}_{0\;r} && \begin{pmatrix} r=1,\,2,\,\ldots\,\infty\\ n=1,\,2,\,\ldots\,\infty\end{pmatrix}\end{aligned}\right\}\tag{11}$$

where in the first term on the right-hand side the lower label p starts from 1 when $n=1$ and from 0 when $n\neq 1$.

In reference 1, recursion formulas were derived for the coefficients $A^{n\;p}_{q\;r}$. In the present report a much more significant approach is introduced. Note that equation (8) can be rewritten in the following form:

$$f_n=\sum_{p=0}^{\infty}(ke^{-y})^{2p+n}\sum_{q=0}^{\substack{n-1\\2p+n-2}}A^n{}_q{}^p\,y^q\tag{12}$$

where

$$A^n{}_q{}^p=\sum_{r=0}^{\infty}A^{n\;\;p}_{q\;p+r}\,k^{2r}$$

In a manner similar to that described in reference 1 for the coefficients $A^{n\;p}_{q\;r}$, recursion formulas can be obtained for the power series $A^n{}_q{}^p$. Indeed, the two types of recursion formulas are intimately connected and, for a given value of p, the one can be obtained from the other by mere inspection. A single recursion formula can be written for the general quantity $A^n{}_q{}^p$ but the resulting expression is cumbersome and serves no practical purpose. It is much more desirable to obtain separate recursion formulas for each value of p. As examples of procedure, the recursion formulas for $p=0$, 1, and 2 are considered in the sections which follow.

RECURSION FORMULA FOR $A^n{}_q{}^0$

With $p=0$, the recursion formula is (compare with eq. (57) of ref. 1)

$$\begin{aligned}2n(q+1)A^{n\;\;0}_{q+1}=&(q+1)(q+2)\delta^{n-2}_{q}A^{n\;\;0}_{q+2}+\\ &\frac{1}{4}n\sum_{q_1=0}^{q}\sum_{m=q-q_1}^{n-2-q_1}(m+1)(n-m-1)A^{n-m-1\;0}_{q_1}A^{m+1\;\;0}_{q-q_1}\end{aligned}$$

$$(n=2,3,\ldots\infty;\;q=0,1,\ldots n-2)\tag{13}$$

where

$$\delta^{n-2}_{q}=\begin{matrix}0 & (q=n-2)\\ 1 & (q\neq n-2)\end{matrix}$$

This recursion formula can be solved, the solution starting with $q=n-2$ and descending towards $q=0$. Thus, for $q=n-2$, equation (13) becomes

$$8(n-1)A^{n\;\;0}_{n-1}=\sum_{m=0}^{n-2}(m+1)(n-m-1)A^{m+1\;0}_{m}A^{n-m-1\;0}_{n-m-2}$$

$$(n=2,3,\ldots\infty)\tag{14}$$

Now, multiply both sides of this equation by r^n where r is a dummy variable and sum from $n=2$ to $n=\infty$. Then

$$8\sum_{n=1}^{\infty}(n-1)A^{n\;\;0}_{n-1}r^n=\sum_{n=2}^{\infty}r^n\sum_{m=0}^{n-2}(m+1)(n-m-1)A^{m+1\;0}_{m}A^{n-m-1\;0}_{n-m-2}\tag{15}$$

Let

$$A_{1\,0}=\sum_{n=1}^{\infty}A^{n\;\;0}_{n-1}\,r^n$$

then

$$A_{1\,0}'=\sum_{n=1}^{\infty}nA^{n\;\;0}_{n-1}\,r^{n-1}$$

and

$$(A_{1\,0}')^2=\sum_{n=2}^{\infty}r^{n-2}\sum_{m=1}^{n-1}m(n-m)A^{m\;\;0}_{m-1}\,A^{n-m\;\;0}_{n-m-1}.$$

By observation, it can be seen that the right-hand side of equation (15) is equal to $(rA_{1\,0}')^2$. Equation (15) can thus be replaced by the following first-order nonlinear differential equation:

$$(rA_{1\,0}')^2-8rA_{1\,0}'+8A_{1\,0}=0\tag{16}$$

The solution of this equation is

$$A_{1\,0}'=c_oe^{\frac{1}{4}rA_{1\,0}'}$$

where c_o is the arbitrary constant of integration. From the definition of $A_{1\,0}$ it follows that, with $r=0$,

$$c_o=A^1{}_0{}^0$$

and hence

$$\frac{A_{1\,0}'}{A^1{}_0{}^0}=e^{\frac{1}{4}rA^1{}_0{}^0\frac{A_{1\,0}'}{A^1{}_0{}^0}}\tag{17}$$

Note that by definition $A^n{}_0{}^0$ involves the coefficients $A^{n\;0}_{0\;r}$ which are ultimately calculated by means of the auxiliary relations, equations (11), engendered by the boundary conditions at the surface of the wavy wall. In solutions of the recursion formulas, therefore, the coefficients $A^n{}_0{}^0$ appear as undetermined quantities.

The expansion of $\dfrac{A_{1\,0}'}{A^1{}_0{}^0}$ in powers of $rA^1{}_0{}^0$ is a nontrivial problem which fortunately can be solved with the aid of Lagrange's investigations on the reversion of power series. In reference 2, Lagrange's problem is illustrated by the following example:

To expand e^{ax} in powers of $y=xe^{bx}$. The result is expressed as

$$e^{ax}=1+ay+\frac{a(a-2b)}{2!}y^2+\frac{a(a-3b)^2}{3!}y^3+\ldots+\frac{a(a-nb)^{n-1}}{n!}y^n+\ldots\tag{18}$$

In particular, with $a=1$, $b=-1$, and $\xi=e^x$, the solution of the equation $\log\xi=y\xi$ or $\xi=e^{y\xi}$ (generally referred to as Eisenstein's problem) is given by the series

$$\xi=\sum_{n=1}^{\infty}\frac{n^{n-2}}{(n-1)!}y^{n-1}$$

Then, if ξ is replaced by $\dfrac{A_{1\,0}'}{A^1{}_0{}^0}$ and y by $\frac{1}{4}rA^1{}_0{}^0$, the solution of equation (17) is

$$\frac{A_{1\,0}'}{A^1{}_0{}^0}=\sum_{n=1}^{\infty}\frac{n^{n-2}}{(n-1)!\,4^{n-1}}(A^1{}_0{}^0)^{n-1}r^{n-1}$$

or

$$\sum_{n=1}^{\infty} n A^{n}_{n-1}{}^{0} r^{n-1} = \sum_{n=1}^{\infty} \frac{n^{n-2}}{(n-1)!\,4^{n-1}} (A^{1}_{0}{}^{0})^{n} r^{n-1}$$

Hence, by equating coefficients of equal powers of r on both sides of this equation,

$$A^{n}_{n-1}{}^{0} = \frac{n^{n-2}}{n!4^{n-1}} (A^{1}_{0}{}^{0})^{n} \quad (n=2, 3, \ldots \infty) \tag{19}$$

which is the solution of the recursion formula, equation (14). The coefficient $A^{n}_{n-1}{}^{0}$ can be considered as the generating function for the set of coefficients $A^{n}_{n-1}{}^{0}_{r}$. Thus, equation (19) can be written as

$$\sum_{r=0}^{\infty} A^{n}_{n-1}{}^{0}_{r} k^{2r} = \frac{n^{n-2}}{n!4^{n-1}} \left(\sum_{r=0}^{\infty} A^{1}_{0}{}^{0}_{r} k^{2r} \right)^{n} \tag{20}$$

Then, by equating coefficients of equal powers of k on both sides of this equation and putting $A^{1}_{0}{}^{0}_{0}=1$ (see boundary condition, eqs. (10)), the following equations are obtained:

$$\left.\begin{aligned}
A^{n}_{n-1}{}^{0}_{0} &= \frac{n^{n-2}}{n!4^{n-1}} \\
A^{n}_{n-1}{}^{0}_{1} &= \frac{n^{n-2}}{(n-1)!4^{n-1}} A^{1}_{0}{}^{0}_{1} \\
A^{n}_{n-1}{}^{0}_{2} &= \frac{n^{n-2}}{(n-1)!4^{n-1}} A^{1}_{0}{}^{0}_{2} + \frac{n^{n-2}}{2!(n-2)!4^{n-1}} (A^{1}_{0}{}^{0}_{1})^{2} \\
&\ldots\ldots
\end{aligned}\right\} \tag{21}$$

Note that from the first of these formulas $A^{2}_{1}{}^{0}_{0}=\frac{1}{8}$ and that from the boundary condition, equations (11), $A^{2}_{0}{}^{0}_{0}=\frac{1}{16}$.

Consider now $q=n-3$. Then equation (13) becomes

$$2n(n-2) A^{n}_{n-2}{}^{0} = (n-1)(n-2) A^{n}_{n-1}{}^{0} + \frac{1}{2} n \sum_{m=2}^{n-1} m(n-m) A^{m}_{m-2}{}^{0} A^{n-m}_{n-m-1}{}^{0} \tag{22}$$

Multiply both sides of this equation by r^{n} and sum from $n=3$ to $n=\infty$. Then

$$2 \sum_{n=2}^{\infty} n(n-2) A^{n}_{n-2}{}^{0} r^{n} = \sum_{n=1}^{\infty} (n-1)(n-2) A^{n}_{n-1}{}^{0} r^{n} + \frac{1}{2} \sum_{n=3}^{\infty} n r^{n} \sum_{m=2}^{n-1} m(n-m) A^{m}_{m-2}{}^{0} A^{n-m}_{n-m-1}{}^{0} \tag{23}$$

Let

$$A_{1\,0} = \sum_{n=1}^{\infty} A^{n}_{n-1}{}^{0} r^{n}$$

and

$$A_{2\,0} = \sum_{n=2}^{\infty} A^{n}_{n-2}{}^{0} r^{n}$$

then

$$A_{1\,0}' A_{2\,0}' = \sum_{n=1}^{\infty} r^{n} \sum_{m=1}^{n} m(n-m+2) A^{m}_{m-1}{}^{0} A^{n-m+2}_{n-m}{}^{0}$$

It can be shown easily that the last term on the right-hand side of equation (23) is equal to $\frac{1}{2} r (r^{2} A_{1\,0}' A_{2\,0}')'$. Hence, equation (23) can be rewritten as the following differential equation:

$$2r^{2} A_{2\,0}'' - 2r A_{2\,0}' = r^{2} A_{1\,0}'' - 2r A_{1\,0}' + 2A_{1\,0} + \frac{1}{2} r (r^{2} A_{1\,0}' A_{2\,0}')' \tag{24}$$

Now, from equation (16), it follows that

$$\left.\begin{aligned}
(rA_{1\,0}')^{2} &= 8\,(rA_{1\,0}' - A_{1\,0}) \\
(rA_{1\,0}')' &= \frac{4A_{1\,0}'}{4 - rA_{1\,0}'} \\
rA_{1\,0}'(rA_{1\,0}')' &= 4rA_{1\,0}''
\end{aligned}\right\} \tag{25}$$

Then equation (24) can be written as

$$(rA_{2\,0}')' - \frac{8 + r(rA_{1\,0}')'}{r(4 - rA_{1\,0}')} (rA_{2\,0}') = \frac{1}{r(4 - rA_{1\,0}')} [2r(rA_{1\,0}')' - 6rA_{1\,0}' + 4A_{1\,0}] \tag{26}$$

Let $rA_{2\,0}' = v \frac{(rA_{1\,0}')^{2}}{4 - rA_{1\,0}'}$, where v is the new dependent variable. Then equation (26) becomes

$$v' = \frac{1}{r(rA_{1\,0}')^{2}} [2r(rA_{1\,0}')' - 6rA_{1\,0}' + 4A_{1\,0}]$$

or with the aid of equations (25),

$$v' = \frac{1}{8} (rA_{1\,0}')'$$

Hence

$$v = \frac{1}{8} rA_{1\,0}' + \frac{1}{8} c_{1}$$

or

$$8rA_{2\,0}' = \frac{(rA_{1\,0}')^{3}}{4 - rA_{1\,0}'} + c_{1} \frac{(rA_{1\,0}')^{2}}{4 - rA_{1\,0}'}$$

The arbitrary constant c_{1} is determined by differentiating this last equation twice and evaluating for $r=0$. Thus,

$$c_{1} = 64 \frac{A^{2}{}_{0}{}^{0}}{(A^{1}{}_{0}{}^{0})^{2}}$$

Finally,

$$A_{2\,0}' = \frac{1}{16} r^{2} [(A_{1\,0}')^{2}]' + 8 \frac{A^{2}{}_{0}{}^{0}}{(A^{1}{}_{0}{}^{0})^{2}} rA_{1\,0}'' \tag{27}$$

Now, from equation (18) with $a=2$, $b=-1$, and $\xi=e^{x}$, it follows that

$$\xi^{2} = 2 \sum_{n=2}^{\infty} \frac{n^{n-3}}{(n-2)!} y^{n-2}$$

Or with

$$\xi=\frac{A_{1\,0}{}'}{A^{1}{}_{0}{}^{0}}\qquad y=\frac{1}{4}\,rA^{1}{}_{0}{}^{0},$$

$$(A_{1\,0}{}')^2=2\sum_{n=2}^{\infty}\frac{n^{n-3}}{(n-2)!4^{n-2}}\,(A^{1}{}_{0}{}^{0})^n r^{n-2}$$

and

$$[(A_{1\,0}{}')^2]'=2\sum_{n=3}^{\infty}\frac{n^{n-3}}{(n-3)!4^{n-2}}\,(A^{1}{}_{0}{}^{0})^n r^{n-3}$$

Thus, equation (27) becomes

$$\sum_{n=2}^{\infty}nA^{n}_{n-2}{}^{0}_{2}\,r^{n-1}=\frac{1}{2}\sum_{n=2}^{\infty}\frac{n^{n-3}}{(n-3)!4^{n-2}}\,(A^{1}{}_{0}{}^{0})^n r^{n-1}+$$
$$8\,\frac{A^{2}{}_{0}{}^{0}}{(A^{1}{}_{0}{}^{0})^2}\sum_{n=2}^{\infty}n(n-1)\,A^{n}_{n-1}{}^{0}_{1}\,r^{n-1}$$

Hence, replacing $A^{n}_{n-1}{}^{0}_{1}$ by its value from equation (19) and equating the coefficients of equal powers of r on both sides of this equation yields

$$A^{n}_{n-2}{}^{0}_{2}=8\,\frac{n^{n-3}}{(n-2)!4^{n-1}}\,(A^{1}{}_{0}{}^{0})^{n-2}\,A^{2}{}_{0}{}^{0}+\frac{1}{2}\,\frac{n^{n-4}}{(n-3)!4^{n-1}}\,(A^{1}{}_{0}{}^{0})^n$$
$$(n=3,4,\;.\;.\;.\;\infty)\qquad(28)$$

the solution of the recursion formula, equation (22). Again, equation (28) can be written as

$$\sum_{r=0}^{\infty}A_{n}{}^{n}_{-2}{}^{0}_{r}\,k^{2r}=8\,\frac{n^{n-3}}{(n-2)!4^{n-1}}\sum_{r=0}^{\infty}A^{2}_{0}{}^{0}_{r}\,k^{2r}\left(\sum_{r=0}^{\infty}A^{1}_{0}{}^{0}_{r}\,k^{2r}\right)^{n-2}+$$
$$\frac{1}{2}\,\frac{n^{n-4}}{(n-3)!4^{n-1}}\left(\sum_{r=0}^{\infty}A^{1}_{0}{}^{0}_{r}\,k^{2r}\right)^n\qquad(29)$$

Then, equating the coefficients of equal powers of k on both sides of this equation and putting $A^{1}_{0}{}^{0}_{0}=1$ and $A^{2}_{0}{}^{0}_{0}=\frac{1}{16}$ yields:

$$\left.\begin{aligned}
&A_{n}{}^{n}_{-2}{}^{0}_{0}=\frac{n^{n-3}}{2(n-2)!4^{n-1}}+\frac{n^{n-4}}{2(n-3)!4^{n-1}}\\
&A_{n}{}^{n}_{-2}{}^{0}_{1}=\frac{8n^{n-3}}{(n-2)!4^{n-1}}\,A^{2}_{0}{}^{0}_{1}+\frac{n^{n-3}}{(n-3)!4^{n-1}}\,A^{1}_{0}{}^{0}_{1}\\
&A_{n}{}^{n}_{-2}{}^{0}_{2}=\frac{8n^{n-3}}{(n-2)!4^{n-1}}\,A^{2}_{0}{}^{0}_{2}+\\
&\qquad\frac{n^{n-3}}{(n-3)!4^{n-1}}\Big[A^{1}_{0}{}^{0}_{2}+8A^{1}_{0}{}^{0}_{1}\,A^{2}_{0}{}^{0}_{1}+\\
&\qquad\frac{1}{2}\,(A^{1}_{0}{}^{0}_{1})^2\Big]+\frac{n^{n-3}}{2(n-4)!4^{n-1}}\,(A^{1}_{0}{}^{0}_{1})^2\\
&.\;\;.\;\;.\;\;.\;\;.
\end{aligned}\right\}\qquad(30)$$

Note that from the first of these formulas $A^{3}_{1}{}^{0}_{0}=\frac{1}{24}$ and that from the boundary conditions, equations (11), $A^{3}_{0}{}^{0}_{0}=\frac{1}{72}$.

The procedure followed in order to obtain equation (28) for $A^{n}_{n-2}{}^{0}_{2}$ is a general one and with very little difficulty other members of the family $A^{n}{}_{q}{}^{0}$ can be obtained. For example,

$$A^{n}_{n-3}{}^{0}_{3}=48\,\frac{n^{n-4}}{(n-3)!4^{n-1}}(A^{1}{}_{0}{}^{0})^{n-3}A^{3}{}_{0}{}^{0}+$$
$$\frac{1}{8}\,\frac{n^{n-4}}{(n-4)!4^{n-1}}(A^{1}{}_{0}{}^{0})^{n-4}[16A^{2}{}_{0}{}^{0}+(A^{1}{}_{0}{}^{0})^2]^2\qquad(31)$$

From this equation it follows that

$$\left.\begin{aligned}
&A_{n}{}^{n}_{-3}{}^{0}_{0}=\frac{2}{3}\,\frac{n^{n-4}}{(n-3)!4^{n-1}}+\frac{1}{2}\,\frac{n^{n-4}}{(n-4)!4^{n-1}}\\
&A_{n}{}^{n}_{-3}{}^{0}_{1}=48\,\frac{n^{n-4}}{(n-3)!4^{n-1}}\,A^{3}_{0}{}^{0}_{1}+8\,\frac{n^{n-4}}{(n-4)!4^{n-1}}\,A^{2}_{0}{}^{0}_{1}+\\
&\qquad\frac{n^{n-4}}{4^{n-1}}\,A^{1}_{0}{}^{0}_{1}\left[\frac{5}{3}\,\frac{1}{(n-4)!}+\frac{1}{2}\,\frac{1}{(n-5)!}\right]\\
&.\;\;.\;\;.\;\;.\;\;.
\end{aligned}\right\}\qquad(32)$$

From the first of these formulas and the boundary conditions, equations (11), $A^{4}_{1}{}^{0}_{0}=\frac{7}{384}$ and $A^{4}_{0}{}^{0}_{0}=\frac{7}{1536}$.

At this point, it is noted that the coefficients of the form $A^{n}_{1}{}^{0}_{0}$ are calculated from the first formula of each set, equations (21), (30), (32), and so forth. A number of this type of coefficient have been evaluated (see ref. 1) by means of the recursion formulas for the coefficients $A^{n}_{q}{}^{p}_{r}$ themselves. They are listed as follows:

$$A^{2}_{1}{}^{0}_{0}=\frac{1}{8}\qquad A^{6}_{1}{}^{0}_{0}=\frac{7\times13}{72\times256}$$

$$A^{3}_{1}{}^{0}_{0}=\frac{1}{24}\qquad A^{7}_{1}{}^{0}_{0}=\frac{13}{18\times256}$$

$$A^{4}_{1}{}^{0}_{0}=\frac{7}{384}\qquad A^{8}_{1}{}^{0}_{0}=\frac{13\times19}{576\times256}$$

$$A^{5}_{1}{}^{0}_{0}=\frac{7}{768}\qquad .\;.\;.\;.\;.$$

A careful examination of these numerical values leads to the general rule,

$$A^{n}_{1}{}^{0}_{0}=\frac{\{3n-5\}}{n!4^{n-1}}\qquad(n=2,3,\;.\;.\;.\;\infty)$$

and from the boundary conditions, equations (11),

$$A^{n}_{0}{}^{0}_{0}=\frac{\{3n-5\}}{nn!4^{n-1}}\qquad(n=2,3,\;.\;.\;.\;\infty)$$

where by definition

$$\{3n-5\}=1\times4\times7\times10\times13\times\;.\;.\;.\;\times(3n-5)$$

In the expression for the local Mach number distribution

evaluated at the crest of the wavy wall ($x=0$, $y=0$), there occurs the following power series:

$$F=\sum_{n=1}^{\infty} nA^{n\,0}_{0\,0}\, k^n$$

or

$$F=k+\sum_{n=2}^{\infty} \frac{\{3n-5\}}{n!\,4^{n-1}}\, k^n$$

This power series can be expressed in the closed form

$$F=2-2\left(1-\frac{3}{4}\,k\right)^{2/3}.$$

The graph of F against k is a semicubical parabola with the cusp point at $k=\frac{4}{3}$ and $F=2$. With the necessary condition that one and only one value of k correspond to a given value of F, the transonic similarity parameter k cannot be greater than 4/3. Moreover, the lower limit of k is zero when the amplitude of the wavy wall is zero but M_∞ is different from unity.

RECURSION FORMULA FOR $A^{n}_{q}{}^{1}$

With $p=1$, the recursion formula for $A^{n}_{q}{}^{1}$ is (compare with eq. (69) of ref. 1)

$$4(n+1)A^{n}_{q}{}^{1}-2(n+2)(q+1)\delta^{n}_{q}A^{n}_{q+1}{}^{1}+(q+1)(q+2)\delta^{n-1,n}_{q}A^{n}_{q+2}{}^{1}=$$

$$-\frac{1}{2}n\delta^{n}_{q}\sum_{q_1=0}^{q}\;\sum_{m=q-q_1}^{n-1-q_1} m(n-m)A^{n-m}_{q_1}{}^{0}A^{m}_{q-q_1}{}^{1}-\frac{1}{2}n(n+1)A^{1}_{0}{}^{0}A^{n+1}_{q}{}^{0}$$

$$(n=1,\,2,\,\ldots\infty;\; q=0,1,\ldots n) \qquad (33)$$

where

$$\delta^{n-1,n}_{q}=\begin{matrix}0\\1\end{matrix}\qquad \begin{matrix}(q=n-1 \text{ or } n)\\(q\neq n-1 \text{ or } n)\end{matrix}$$

$$\delta^{n}_{q}=\begin{matrix}0\\1\end{matrix}\qquad \begin{matrix}(q=n)\\(q\neq n)\end{matrix}$$

The solution of this recursion formula proceeds as in the case $p=0$, the solution starting with $q=n$ and descending towards $q=0$. Thus, for $q=n$, equation (33) becomes

$$A^{n}_{n}{}^{1}=-\frac{1}{8}n\,A^{1}_{0}{}^{0}\,A^{n+1}_{n}{}^{0}\qquad (n=1,\,2,\ldots\infty)$$

or

$$A^{n-1}_{n-1}{}^{1}=-\frac{1}{8}(n-1)A^{1}_{0}{}^{0}\,A^{n}_{n-1}{}^{0}\qquad (n=2,\,3,\ldots\infty)$$

Hence, inserting the expression for $A^{n}_{n-1}{}^{0}$ given by equation (19) gives

$$A^{n-1}_{n-1}{}^{1}=-\frac{1}{8}\,\frac{n^{n-3}}{(n-2)!\,4^{n-1}}\,(A^{1}_{0}{}^{0})^{n+1}\qquad (n=2,\,3,\,\ldots\,\infty)\quad (34)$$

or

$$\sum_{r=0}^{\infty} A^{n-1}_{n-1}{}^{1}_{r+1}\, k^{2r}=-\frac{1}{8}\,\frac{n^{n-3}}{(n-2)!\,4^{n-1}}\left(\sum_{r=0}^{\infty} A^{1}_{0}{}^{0}_{r}\, k^{2r}\right)^{n+1}$$

From this relation, by equating coefficients of equal powers of k on both sides of the equation, the following equations are obtained:

$$\left.\begin{aligned}
A^{n-1}_{n-1}{}^{1}_{1}&=-\frac{1}{8}\,\frac{n^{n-3}}{(n-2)!\,4^{n-1}}\\
A^{n-1}_{n-1}{}^{1}_{2}&=-\frac{3}{8}\,\frac{n^{n-3}}{(n-2)!\,4^{n-1}}\,A^{1}_{0}{}^{0}_{1}-\frac{1}{8}\,\frac{n^{n-3}}{(n-3)!\,4^{n-1}}\,A^{1}_{0}{}^{0}_{1}\\
A^{n-1}_{n-1}{}^{1}_{3}&=-\frac{3}{8}\,\frac{n^{n-3}}{(n-2)!\,4^{n-1}}\,[A^{1}_{0}{}^{0}_{2}+(A^{1}_{0}{}^{0}_{1})^2]-\\
&\qquad\frac{1}{8}\,\frac{n^{n-3}}{(n-3)!\,4^{n-1}}\,[A^{1}_{0}{}^{0}_{2}+3(A^{1}_{0}{}^{0}_{1})^2]-\\
&\qquad\frac{1}{16}\,\frac{n^{n-3}}{(n-4)!\,4^{n-1}}\,(A^{1}_{0}{}^{0}_{1})^2\\
\ldots\ldots&\qquad\qquad (n=2,\,3,\,\ldots\,\infty)
\end{aligned}\right\}\quad (35)$$

Consider now $q=n-1$; equation (33) becomes

$$4(n+1)A^{n}_{n-1}{}^{1}=2n(n+2)A^{n}_{n}{}^{1}-$$

$$\frac{1}{2}n\sum_{m=0}^{n-1}(m+1)(n-m-1)A^{m+1}_{m}{}^{0}\,A^{n-m-1}_{n-m-1}{}^{1}-$$

$$\frac{1}{2}\,n(n+1)A^{1}_{0}{}^{0}\,A^{n+1}_{n-1}{}^{0}\qquad (n=1,2,\,\ldots\,\infty)$$

Multiply both sides of this equation by r^n and sum from $n=1$ to $n=\infty$. Then

$$4\sum_{n=1}^{\infty}(n+1)A^{n}_{n-1}{}^{1}\,r^n=2\sum_{n=1}^{\infty}n(n+2)A^{n}_{n}{}^{1}\,r^n-$$

$$\frac{1}{2}\sum_{n=1}^{\infty}nr^n\sum_{m=0}^{n-1}(m+1)(n-m-1)A^{m+1}_{m}{}^{0}\,A^{n-m-1}_{n-m-1}{}^{1}-$$

$$\frac{1}{2}\,A^{1}_{0}{}^{0}\sum_{n=1}^{\infty}n(n+1)A^{n+1}_{n-1}{}^{0}\,r^n \qquad (36)$$

Let

$$A_{0\,1}=\sum_{n=1}^{\infty}A^{n}_{n}{}^{1}\,r^n \qquad A_{1\,1}=\sum_{n=1}^{\infty}A^{n}_{n-1}{}^{1}\,r^n$$

$$A_{1\,0}=\sum_{n=1}^{\infty}A^{n}_{n-1}{}^{0}\,r^n \qquad A_{2\,0}=\sum_{n=2}^{\infty}A^{n}_{n-2}{}^{0}\,r^n$$

Then it can easily be shown that the second term on the right-hand side of equation (36) is $-\frac{1}{2}\,r(r^2A_{0\,1}'A_{1\,0}')'$ and that equation (36) can be replaced by

$$4(rA_{1\,1})'=2r^2A_{0\,1}''+6rA_{0\,1}'-\frac{1}{2}r(r^2A_{0\,1}'A_{1\,0}')'-\frac{1}{2}\,rA^{1}_{0}{}^{0}\,A_{2\,0}'' \qquad (37)$$

Now,

$$A_{0\,1}=-\frac{1}{8}\,A^{1}_{0}{}^{0}\sum_{n=1}^{\infty}(n-1)A^{n}_{n-1}{}^{0}\,r^{n-1}$$

or, with the aid of equations (25),

$$rA_{0\,1}=-\frac{1}{64}\,A^{1}_{0}{}^{0}(rA_{1\,0}')^2$$

Hence, from the use of equations (25) and equation (27) for $A_{2\,0}'$ it follows, after routine calculations, that the integral of equation (37) is

$$rA_{1\,1}=\frac{1}{128}A^{1}_{0}{}^{0}\left\{-\frac{1}{2}(rA_{1\,0}')^2-\frac{1}{4}(rA_{1\,0}')^3-\frac{1}{64}(rA_{1\,0}')^4+(c_1+4)\left[\frac{1}{4}(rA_{1\,0}')^2-2r^2A_{1\,0}''\right]\right\}$$

where

$$c_1=64\frac{A^{2}_{0}{}^{0}}{(A^{1}_{0}{}^{0})^2}$$

Now from equation (18),

$$(rA_{1\,0}')^m=m\sum_{n=m}^{\infty}\frac{n^{n-m-1}}{(n-m)!4^{n-m}}(A^{1}_{0}{}^{0})^n r^n \tag{38}$$

Then

$$\sum_{n=2}^{\infty}A^{n-1\,1}_{n-2}r^n=\frac{1}{128}(A^{1}_{0}{}^{0})^{n+1}\left\{-\sum_{n=2}^{\infty}\frac{n^{n-3}}{(n-2)!4^{n-2}}r^n-\frac{3}{4}\sum_{n=3}^{\infty}\frac{n^{n-4}}{(n-3)!4^{n-3}}r^n-\frac{1}{16}\sum_{n=4}^{\infty}\frac{n^{n-5}}{(n-4)!4^{n-4}}r^n+\right.$$
$$\left.(c_1+4)\left[\frac{1}{2}\sum_{n=2}^{\infty}\frac{n^{n-3}}{(n-2)!4^{n-2}}r^n-2\sum_{n=2}^{\infty}\frac{n^{n-2}}{(n-2)!4^{n-1}}r^n\right]\right\} \tag{39}$$

Equating coefficients of equal powers of r on both sides of this equation gives

$$A^{n-1\,1}_{n-2}=-\frac{1}{64}\left\{4[2+(c_1+4)]\frac{1}{(n-2)!}+14[2+(c_1+4)]\frac{1}{(n-3)!}+2[5+4(c_1+4)]\frac{1}{(n-4)!}+(c_1+4)\frac{1}{(n-5)!}\right\}\frac{n^{n-5}}{4^{n-1}}(A^{1}_{0}{}^{0})^{n+1}$$
$$(n=2,3,\ldots\infty) \tag{40}$$

From this equation the expressions for all the coefficients of the type $A^{n-1\,1}_{n-2\,r+1}$ with $n=2, 3, \ldots \infty$ and $r=0, 1, \ldots \infty$ can be obtained. Thus, for example,

$$\left.\begin{aligned}
A^{n-1\,1}_{n-2\,1}&=-\frac{1}{32}\frac{n^{n-5}}{4^{n-1}}\left[\frac{20}{(n-2)!}+\frac{70}{(n-3)!}+\frac{37}{(n-4)!}+\frac{4}{(n-5)!}\right]\\
A^{n-1\,1}_{n-2\,2}&=-\frac{1}{32}\frac{n^{n-5}}{4^{n-1}}\left\{[(5n+1)A^{1\,0}_{0\,1}+32A^{2\,0}_{0\,1}]\frac{4}{(n-2)!}+[(5n+1)A^{1\,0}_{0\,1}+32A^{2\,0}_{0\,1}]\frac{14}{(n-3)!}+[(37n+5)A^{1\,0}_{0\,1}+\right.\\
&\qquad\left.256A^{2\,0}_{0\,1}]\frac{1}{(n-4)!}+(nA^{1\,0}_{0\,1}+8A^{2\,0}_{0\,1})\frac{1}{(n-5)!}\right\}\\
&\cdots\cdots
\end{aligned}\right\} \tag{41}$$

For $q=n-2$, equation (33) becomes

$$4(n+1)A^{n\,1}_{n-2}=2(n-1)(n+2)A^{n\,1}_{n-1}-n(n-1)A^{n\,1}_{n}-\frac{1}{2}n\sum_{m=0}^{n-2}[(m+2)(n-m-2)A^{m+2\,0}_{m}A^{n-m-2\,1}_{n-m-2}+$$
$$(m+1)(n-m-1)A^{m+1\,0}_{m}A^{n-m-1\,1}_{n-m-2}]-\frac{1}{2}n(n+1)A^{1}_{0}{}^{0}A^{n+1\,0}_{n-2} \tag{42}$$

The solution of this recursion formula follows along the same lines as that for equation (36) and leads to the following result:

$$A^{n-1\,1}_{n-3}=-\frac{1}{512}\left\{\left[\frac{162}{(n-3)!}+\frac{414}{(n-4)!}+\frac{205}{(n-5)!}+\frac{47}{2}\frac{1}{(n-6)!}\right]\frac{n^{n-7}}{4^{n-2}}+(c_1+4)\left[\frac{21}{(n-3)!}+\frac{24}{(n-4)!}+\frac{9}{2}\frac{1}{(n-5)!}\right]\frac{n^{n-5}}{4^{n-2}}+\right.$$
$$\left.12\times128\frac{A^{3}_{0}{}^{0}}{(A^{1}_{0}{}^{0})^3}\left[\frac{1}{(n-3)!}+\frac{1}{2}\frac{1}{(n-4)!}\right]\frac{n^{n-4}}{4^{n-2}}+\frac{1}{8}(c_1+4)^2\left[\frac{3}{(n-4)!}+\frac{1}{(n-5)!}\right]\frac{n^{n-4}}{4^{n-2}}\right\}(A^{1}_{0}{}^{0})^{n+1}$$
$$(n=3,4,\ldots\infty) \tag{43}$$

From this equation there follow, in the usual manner, expressions for coefficients of the type $A^{n-1\,1}_{n-3\,r+1}$. Thus, for example, with $r=0$,

$$A^{n-1\,1}_{n-3\,1}=-\left[\left(\frac{81}{64}+\frac{21}{16}n^2+\frac{1}{6}n^3\right)\frac{1}{(n-3)!}+\left(\frac{207}{64}+\frac{3}{2}n^2+\frac{13}{48}n^3\right)\frac{1}{(n-4)!}+\left(\frac{205}{128}+\frac{9}{32}n^2+\frac{1}{16}n^3\right)\frac{1}{(n-5)!}+\frac{47}{256}\frac{1}{(n-6)!}\right]\frac{n^{n-7}}{4^{n-1}}$$
$$\cdots\cdots \tag{44}$$

RECURSION FORMULA FOR $A^{n}{}_{q}{}^{2}$

With $p=2$, the recursion formula for $A^{n}{}_{q}{}^{2}$ is

$$8(n+2)A^{n}{}_{q}{}^{2}-2(n+4)(q+1)\delta^{n+2}_{q}A^{n}_{q+1}{}^{2}+ (q+1)(q+2)\delta^{n+1,n+2}_{q}A^{n}_{q+2}{}^{2}$$

$$=-\frac{1}{2}n\delta^{n+2}_{q}\,\delta^{n-1,n,n+1}_{q_1}\,\delta^{0,1,2}_{m}\sum_{q_1=0}^{q}\sum_{m=q-q_1}^{n+1-q_1}(m-2)(n-m+2)A^{n-m+2}_{q_1}{}^{0}\,A^{m-2}_{q-q_1}{}^{2}-\frac{1}{4}n\delta^{n+1,n+2}_{q}\,\delta^{n}_{q_1}\sum_{q_1=0}^{q}\sum_{m=q-q_1}^{n-q_1}m(n-m)A^{n-m}_{q_1}{}^{1}\,A^{m}_{q-q_1}{}^{1}-\frac{1}{2}n(n+1)(\delta^{n+2}_{q}\,A^{1}{}_{0}{}^{0}\,A^{n+1}_{q}{}^{1}+\delta^{n+1,n+2}_{q}\,A^{1}{}_{0}{}^{1}\,A^{n+1}_{q}{}^{0}+\delta^{0,n+2}_{q}\,A^{1}{}_{1}{}^{1}\,A^{n+1}_{q-1}{}^{0})-n(n+2)(\delta^{n+2}_{q}\,A^{2}{}_{0}{}^{0}\,A^{n+2}_{q}{}^{0}+\delta^{0}_{q}\,A^{2}{}_{1}{}^{0}\,A^{n+2}_{q-1}{}^{0})$$

$$(n=1,2,\ldots\infty;\ q=0,1,\ldots n+2) \qquad (45)$$

where δ is defined in the usual manner. The solution of this recursion formula proceeds as in the previous cases, starting with $q=n+2$ and descending towards $q=0$. Thus. for $q=n+2$, equation (45) becomes:

$$A^{n}_{n+2}{}^{2}=-\frac{1}{8}nA^{2}{}_{1}{}^{0}\,A^{n+2}_{n+1}{}^{0} \qquad (n=1,2,\ldots\infty)$$

or

$$A^{n-2}_{n}{}^{2}=-\frac{1}{8}(n-2)A^{2}{}_{1}{}^{0}\,A^{n}_{n-1}{}^{0} \qquad (n=3,4,\ldots\infty)$$

Then, from equation (19),

$$A^{n-2}_{n}{}^{2}=-\frac{1}{64}(n-2)\frac{n^{n-2}}{n!4^{n-1}}(A^{1}{}_{0}{}^{0})^{n+2}$$

$$(n=3,4,\ldots\infty) \qquad (46)$$

or

$$\sum_{r=0}^{\infty}A^{n-2}_{n}{}^{2}_{r+2}k^{2r}=-\frac{(n-2)n^{n-2}}{n!4^{n+2}}\left(\sum_{r=0}^{\infty}A^{1}_{0}{}^{0}_{r}\,k^{2r}\right)^{n+2}$$

By equating coefficients of equal powers of k on both sides of this equation, the following equations are obtained:

$$\left.\begin{aligned}
A^{n-2}_{n}{}^{2}_{2}&=-\frac{(n-2)n^{n-2}}{n!4^{n+2}}\\
A^{n-2}_{n}{}^{2}_{3}&=-\frac{(n^2-4)n^{n-2}}{n!4^{n+2}}A^{1}_{0}{}^{0}_{1}\\
A^{n-2}_{n}{}^{2}_{4}&=-\frac{(n^2-4)n^{n-2}}{n!4^{n+2}}\left[A^{1}_{0}{}^{0}_{2}+\frac{1}{2}(n+1)(A^{1}_{0}{}^{0}_{1})^2\right]\\
&\ldots\ldots
\end{aligned}\right\} \qquad (47)$$

Consider now $q=n+1$; equation (45) becomes

$$8(n+2)A^{n}_{n+1}{}^{2}=2(n+2)(n+4)A^{n}_{n+2}{}^{2}-\frac{1}{2}n\sum_{m=0}^{n-2}(m+1)(n-m-1)A^{m+1}_{m}{}^{0}A^{n-m-1}_{n-m+1}{}^{2}-\frac{1}{2}n(n+1)(A^{1}{}_{0}{}^{0}A^{n+1}_{n+1}{}^{1}+A^{1}{}_{1}{}^{1}A^{n+1}_{n}{}^{0})-n(n+2)(A^{2}{}_{0}{}^{0}A^{n+2}_{n+1}{}^{0}+A^{2}{}_{1}{}^{0}A^{n+2}_{n}{}^{0})$$

$$(n=1,2,\ldots\infty) \qquad (48)$$

With the introduction of

$$A_{-2\,2}=\sum_{n=1}^{\infty}A^{n}_{n+2}{}^{2}\,r^{n} \qquad A_{2\,0}=\sum_{n=2}^{\infty}A^{n}_{n-2}{}^{0}\,r^{n}$$

$$A_{-1\,2}=\sum_{n=1}^{\infty}A^{n}_{n+1}{}^{2}\,r^{n} \qquad A_{0\,1}=\sum_{n=1}^{\infty}A^{n}{}_{n}{}^{1}\,r^{n}$$

$$A_{1\,0}=\sum_{n=1}^{\infty}A^{n}_{n-1}{}^{0}\,r^{n} \qquad \ldots\ldots$$

equation (48) is replaced by the following ordinary linear differential equation:

$$8(r^2A_{-1\,2})'=2r^2(rA_{-2\,2}')'+12r(rA_{-2\,2}')+16rA_{-2\,2}-\frac{1}{2}r^2(r^2A_{1\,0}'A_{-2\,2}')-\frac{1}{2}A^{1}{}_{0}{}^{0}r^2A_{0\,1}''-\frac{1}{2}A^{1}{}_{1}{}^{1}r^2A_{1\,0}''-A^{2}{}_{0}{}^{0}\,rA_{1\,0}''+A^{2}{}_{0}{}^{0}A_{1\,0}'-A^{2}{}_{1}{}^{0}\,rA_{2\,0}''+A^{2}{}_{1}{}^{0}A_{2\,0}'-A^{1}{}_{0}{}^{0}A^{2}{}_{0}{}^{0} \qquad (49)$$

By repeated use of equations (25), the solution of this differential equation is found to be as follows:

$$8r^2A_{-1\,2}=(A^{1}{}_{0}{}^{0})^2\left\{\frac{3}{32}(rA_{1\,0}')-\frac{1}{32}(rA_{1\,0}')^2-\frac{1}{768}(rA_{1\,0}')^3-\frac{1}{2048}(rA_{1\,0}')^4+\frac{1}{16}r^2A_{1\,0}''+\frac{1}{8}A_{2\,0}+\frac{1}{64}(c_1+4)\left[\frac{1}{8}(rA_{1\,0}')^2-r^2A_{1\,0}''\right]\right\}-\frac{3}{32}r(A^{1}{}_{0}{}^{0})^3+A^{2}{}_{0}{}^{0}\left[A_{1\,0}-\frac{1}{8}(rA_{1\,0}')^2\right]-rA^{1}{}_{0}{}^{0}A^{2}{}_{0}{}^{0} \qquad (50)$$

Finally, with the aid of equation (38) and the definition of $A_{-1\,2}$,

$$A^{n-2}_{n-1}{}^{2}=\left\{\frac{1}{256}\left[\frac{1}{(n-1)!}-\frac{1}{(n-2)!}+\frac{2}{(n-3)!}\right]\frac{n^{n-3}}{4^{n-1}}-\frac{1}{64}\frac{n^{n-5}}{(n-4)!4^{n-1}}+\frac{1}{512}(c_1+4)\left[\frac{1}{(n-1)!}-\frac{1}{(n-2)!}\right]\frac{n^{n-2}}{4^{n-1}}\right\}(A^{1}{}_{0}{}^{0})^{n+2}$$

$$\left(n=3,4,\ldots\infty;\ c_1=64\frac{A^{2}{}_{0}{}^{0}}{(A^{1}{}_{0}{}^{0})^2}\right) \qquad (51)$$

From this equation, there follows in the usual manner formulas for the coefficients of the type $A^{n-2}_{n-1}{}^{2}_{r+2}$. Thus, for $r=0$,

$$A^{n-2}_{n-1}{}^{2}_{2}=\left[\frac{1}{(n-1)!}-\frac{1}{(n-2)!}+\frac{2}{(n-3)!}\right]\frac{n^{n-3}}{4^{n+3}}-\frac{n^{n-5}}{(n-4)!4^{n+2}}+\left[\frac{1}{(n-1)!}-\frac{1}{(n-2)!}\right]\frac{n^{n-2}}{4^{n+2}} \qquad (n=3,4,\ldots\infty)$$

DISCUSSION OF CONVERGENCE OF SMOOTH TYPE OF POTENTIAL FLOW PAST WAVY WALL

In the preceding sections, a number of examples of recursion formulas and their solutions have been given in considerable detail. The purpose of this exposition is threefold:

First, to show the inherent elegance of the method of integration in series although the equations concerned are nonlinear in character; second, to present a type of analysis which may be useful in other problems involving nonlinear differential equations; and third, to indicate that an analytical proof of convergence may ultimately be obtained by careful examination of the recursion formulas for the quantities $A^{n}{}_{q}^{p}$ and their solutions. One example is the obtaining of the general expression for the coefficients $A^{n}{}_{1}^{0}{}_{0}$ and the subsequent conclusion that $k\leqq\frac{4}{3}$.

In actual practice it has been found more convenient to evaluate the coefficients $A^{n}{}_{q}^{p}{}_{r}$ from their separate recursion formulas rather than to derive the general formulas. The appendix contains the exact numerical values of the coefficients necessary for the development of the functions f_n to the eighth power in the transonic similarity parameter k. These coefficients are utilized to demonstrate numerically the test for covergence of the smooth symmetrical type of potential flow (eq. (3)) assumed in this report. Thus, when the form of f_n given by equation (8) is inserted into equation (5) for the local Mach number distribution and evaluated at the surface of the sinusoidal wall ($y=0$), the following result is obtained:

$$\frac{M^2-1}{1-M_{\infty}{}^2}=-1+\sum_{n=1}^{\infty} k^n \sum_{m=0}^{\left[\frac{n}{2}\right]}(n-2m)\cos(n-2m)x\sum_{p=0}^{m} A^{n}{}_{0}^{-2m}{}_{m}^{p} \tag{52}$$

where $\left[\frac{n}{2}\right]$ denotes the integral part of $n/2$. At the crest of the wavy wall ($x=0$), the point of maximum fluid velocity, with the numerical values for the coefficients inserted, equation (52) becomes:

$$\frac{M^2-1}{1-M_{\infty}{}^2}+1=k+\frac{1}{8}k^2+\frac{25}{384}k^3+\frac{337}{9216}k^4+\frac{4043}{576\times256}k^5+\frac{359381}{270\times256^2}k^6+\frac{7326757}{6480\times256^3}k^7+\frac{81688733}{86400\times256^3}k^8+\ldots \tag{53}$$

The critical value of k (that is, when $M=1$) calculated from this equation is

$$k_{cr}=0.83244$$

(Note that in ref. 1 the value $k_{cr}=0.83770$ corresponds to the first six terms of eq. (53).) Consider now the infinite series

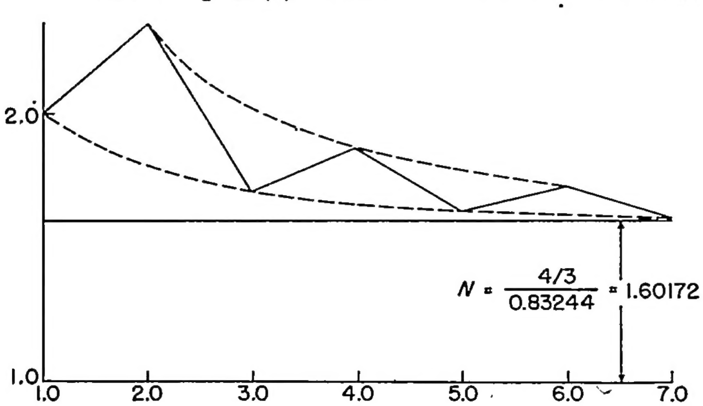

FIGURE 2.—Numerical test of convergence.

$$\sum_{n=1}^{\infty} A^{n}{}_{0}^{0}{}_{0} k^n$$

where

$$A^{n}{}_{0}^{0}{}_{0}=\frac{\{3n-5\}}{nn!\,4^{n-1}} \qquad (n=2, 3, \ldots \infty)$$

The Cauchy ratio test $R_{1n}=\frac{A^{n}{}_{0}^{0}{}_{0}}{A^{n+1}{}_{0}^{0}{}_{0}}$ yields in the limit $n\rightarrow\infty$ the result that the radius of convergence R_1, is equal to or less than 4/3. If the corresponding ratio R_{2n} are formed for the right-hand side of equation (53) and the quotient $N_n=\frac{R_{1n}}{R_{2n}}$ is calculated, the resulting sequence of numbers is as follows:

n	R_{1n}	R_{2n}	N_n
1	16.0	8.0	2.0
2	4.5	1.92	2.34375
3	3.04762	1.78041	1.71175
4	2.5	1.33368	1.87472
5	2.21516	1.35000	1.64104
6	2.04167	1.17721	1.73432
7	1.92481	1.19588	1.60950

The noteworthy feature of this table is that, although R_{1n} (and presumably R_{2n}) is converging quite slowly toward $R_1=\frac{4}{3}$ (and R_2), the quotient N_n exhibits a strong tendency to approach an asymptotic value N for a relatively low value of n. Figure 2 shows this tendency in a graphic manner. The apparent asymptote represented by the straight line is the ratio of 4/3, the limit of R_{1n} as $n\rightarrow\infty$, and of 0.83244, the critical value of k. Certainly, the rapid approach of the lower dotted curve toward the apparent asymptote and the decreasing oscillations represented by the upper dotted curve indicate that the critical value of k is the radius of convergence R_2 of the power series on the right-hand side of equation (53). Thus, the critical value of the stream Mach number marks the limit of convergence of the smooth symmetrical type of potential flow assumed. The ability to approximate closely the limiting value N is a matter of luck; namely, the choice of the known comparison series. Once, however, the proper comparison series has been selected and the approach to an asymptote indicated, there can be no question of the meaningfulness of the approximate value of N obtained. It may be that one would like to extend figure 2 to $n=9$. This extension would entail the forbidding calculation of an additional 185 coefficients $A^{n}{}_{q}^{p}{}_{r}$. The result presumably would be to decrease slightly the approximate critical value of k and thereby raise slightly the straight-line asymptote of figure 2. This extension of figure 2 would show still more convincingly the approach to an asymptote and the dying-out of the oscillations. Perhaps more important still, figure 2 definitely shows that conclusions based on less than six or eight terms are mere speculations in this field.

CONCLUDING REMARKS

If the numerical test of convergence presented is acceptable, the conclusion to be drawn is that smooth symmetrical potential flow past the wavy wall exists only for the purely subsonic range. Moreover no such flow can possibly represent the transonic or mixed type for which a local region of supersonic flow near the solid boundary is imbedded in the

otherwise subsonic stream. It follows as a corollary that the transonic or mixed type of flow past the wavy wall is necessarily an asymmetric one. This asymmetry in the flow pattern entails a resistance usually defined as wave drag. As shown by experimental observations, the shock wave associated with wave drag closes the downstream portion of the local supersonic zone.

As a final remark—although the analysis and conclusions of the present work refer directly to the wavy wall, the suggested result that the critical stream Mach number marks the limit of smooth potential flow very likely applies to other boundaries. This conclusion is based on the idea that the gradual transition from a purely sinusoidal wall to a boundary composed of a single bump, say, introduces no essential changes in the analysis.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., *May 6, 1952.*

APPENDIX

NUMERICAL VALUES OF THE COEFFICIENTS $A^{n\,p}_{q\,r}$

Coefficients for f_1:

$$A^{1\,0}_{0\,0}=1$$

$$A^{1\,0}_{0\,1}=\frac{11}{256}$$

$$A^{1\,0}_{0\,2}=\frac{1861}{3\times256^2}$$

$$A^{1\,0}_{0\,3}=\frac{4896755}{81\times256^3}$$

$$A^{1\,1}_{0\,1}=-\frac{5}{256}$$

$$A^{1\,1}_{0\,2}=-\frac{2765}{9\times256^2}$$

$$A^{1\,1}_{0\,3}=-\frac{184345}{1728\times256^2}$$

$$A^{1\,1}_{1\,1}=-\frac{1}{64}$$

$$A^{1\,1}_{1\,2}=-\frac{33}{64\times256}$$

$$A^{1\,1}_{1\,3}=-\frac{139}{4\times256^2}$$

$$A^{1\,2}_{0\,0}=\frac{65}{576\times256}$$

$$A^{1\,2}_{0\,3}=\frac{26435}{81\times256^3}$$

$$A^{1\,2}_{1\,2}=-\frac{7}{72\times256}$$

$$A^{1\,2}_{1\,3}=-\frac{53995}{1728\times256^2}$$

$$A^{1\,2}_{2\,2}=-\frac{9}{32\times256}$$

$$A^{1\,2}_{2\,3}=-\frac{2765}{96\times256^2}$$

$$A^{1\,2}_{3\,2}=-\frac{1}{8\times256}$$

$$A^{1\,2}_{3\,3}=-\frac{55}{8\times256^2}$$

$$A^{1\,3}_{0\,3}=\frac{3385}{1152\times256^2}$$

$$A^{1\,3}_{1\,3}=\frac{3907}{432\times256^2}$$

$$A^{1\,3}_{2\,3}=\frac{23}{12\times256^2}$$

$$A^{1\,3}_{3\,3}=-\frac{583}{54\times256^2}$$

$$A^{1\,3}_{4\,3}=-\frac{119}{12\times256^2}$$

$$A^{1\,3}_{5\,3}=-\frac{1}{96\times256}$$

Coefficients for f_2:

$$A^{2\,0}_{0\,0}=\frac{1}{16}$$

$$A^{2\,0}_{0\,1}=\frac{419}{72\times256}$$

$$A^{2\,0}_{0\,2}=\frac{234215}{432\times256^2}$$

$$A^{2\,0}_{0\,3}=\frac{42791533}{648\times256^3}$$

$$A^{2\,0}_{1\,0}=\frac{1}{8}$$

$$A^{2\,0}_{1\,1}=\frac{11}{1024}$$

$$A^{2\,0}_{1\,2}=\frac{4085}{24\times256^2}$$

$$A^{2\,0}_{1\,3}=\frac{21287}{324\times256^2}$$

$$A^{2\,1}_{0\,1}=-\frac{125}{36\times256}$$

$$A^{2\,1}_{0\,2}=-\frac{45895}{108\times256^2}$$

$$A^{2\,1}_{0\,3}=-\frac{4210409}{81\times256^3}$$

$$A^{2\,1}_{1\,1}=-\frac{5}{256}$$

$$A^{2\,1}_{1\,2}=-\frac{815}{576\times256}$$

$$A^{2\,1}_{1\,3}=-\frac{456185}{3456\times256^2}$$

$$A^{2\,1}_{2\,1}=-\frac{1}{128}$$

$$A^{2\,1}_{2\,2}=-\frac{11}{32\times256}$$

$$A^{2\,1}_{2\,3}=-\frac{4811}{192\times256^2}$$

$$A^{2\,2}_{0\,2}=\frac{12245}{144\times256^2}$$

$$A^{2\,2}_{0\,3}=\frac{35023583}{1620\times256^3}$$

$$A^{2\,2}_{1\,2}=\frac{6245}{72\times256^2}$$

$$A^{2\,2}_{1\,3}=\frac{104063}{90\times256^3}$$

$$A^{2\,2}_{2\,2}=-\frac{79}{3\times256^2}$$

$$A^{2\,2}_{2\,3}=-\frac{167}{4\times256^2}$$

$$A^{2\,2}_{3\,2}=-\frac{47}{192\times256}$$

$$A^{2\,2}_{3\,3}=-\frac{24199}{864\times256^2}$$

$$A^{2\,2}_{4\,2}=-\frac{1}{12\times256}$$

$$A^{2\,2}_{4\,3}=-\frac{11}{2\times256^2}$$

$$A^{2\,3}_{0\,3}=-\frac{17999}{90\times256^3}$$

$$A^{2\,3}_{1\,3}=\frac{72577}{5760\times256^2}$$

$$A^{2\,3}_{2\,3}=\frac{45409}{2160\times256^2}$$

$$A^{2\,3}_{3\,3}=\frac{3515}{864\times256^2}$$

$$A^{2\,3}_{4\,3}=-\frac{10229}{864\times256^2}$$

$$A^{2\,3}_{5\,3}=-\frac{73}{8\times256^2}$$

$$A^{2\,3}_{6\,3}=-\frac{25}{12\times256^2}$$

Coefficients for f_3:

$$A^{3\,0}_{0\,0}=\frac{1}{72}$$

$$A^{3\,0}_{0\,1}=\frac{23603}{27\times256^2}$$

$$A^{3\,0}_{0\,2}=\frac{9843883}{81\times256^3}$$

$$A^{3\,0}_{1\,0}=\frac{1}{24}$$

$$A^{3\,0}_{1\,1}=\frac{259}{72\times256}$$

$$A^{3\,0}_{1\,2}=\frac{149317}{432\times256^2}$$

$$A^{3\,0}_{2\,0}=\frac{1}{32}$$

$$A^{3\,0}_{2\,1}=\frac{33}{32\times256}$$

$$A^{3\,0}_{2\,2}=\frac{139}{2\times256^2}$$

$$A^{3\,1}_{0\,1}=-\frac{1765}{3\times256^2}$$

$$A^{3\,1}_{0\,2}=-\frac{1351393}{2880\times256^2}$$

$$A^{3\,1}_{1\,1}=-\frac{155}{32\times256}$$

$$A^{3\,1}_{1\,2}=-\frac{30365}{48\times256^2}$$

$$A^{3\,1}_{2\,1}=-\frac{117}{32\times256}$$

$$A^{3\,1}_{2\,2}=-\frac{29545}{96\times256^2}$$

$$A^{3\,1}_{3\,1}=-\frac{1}{256}$$

$$A^{3\,1}_{3\,2}=-\frac{55}{256^2}$$

$$A^{3\,2}_{0\,2}=\frac{475823}{15\times256^3}$$

$$A^{3\,2}_{1\,2}=\frac{74353}{320\times256^2}$$

$$A^{3\,2}_{2\,2}=\frac{3083}{24\times256^2}$$

$$A^{3\,2}_{3\,2}=-\frac{41}{2\times256^2}$$

$$A^{3\,2}_{4\,2}=-\frac{1785}{40\times256^2}$$

$$A^{3\,2}_{5\,2}=-\frac{25}{2\times256^2}$$

Coefficients for f_4:

$$A^{4\,0}_{0\,0}=\frac{7}{6\times256}$$

$$A^{4\,0}_{0\,1}=\frac{390547}{720\times256^2}$$

$$A^{4\,0}_{0\,2}=\frac{1704729613}{16200\times256^3}$$

$$A^{4\,0}_{1\,0}=\frac{7}{384}$$

$$A^{4\,0}_{1\,1}=\frac{32947}{36\times256^2}$$

$$A^{4\,0}_{1\,2}=\frac{20527213}{162\times256^3}$$

$$A^{4\,0}_{2\,0}=\frac{3}{128}$$

$$A^{4\,0}_{2\,1}=\frac{617}{288\times256}$$

$$A^{4\,0}_{2\,2}=\frac{369587}{1728\times256^2}$$

$$A^{4\,0}_{3\,0}=\frac{1}{96}$$

$$A^{4\,0}_{3\,1}=\frac{11}{24\times256}$$

$$A^{4\,0}_{3\,2}=\frac{4811}{144\times256^2}$$

$$A^{4\,1}_{0\,1}=-\frac{35251}{90\times256^3}$$

$$A^{4\,1}_{0\,2}=-\frac{80872738}{675\times256^3}$$

$$A^{4\,1}_{1\,1}=-\frac{2465}{576\times256}$$

$$A^{4\,1}_{1\,2}=-\frac{30596538}{135\times256^3}$$

$$A^{4\,1}_{2\,1}=-\frac{455}{96\times256}$$

$$A^{4\,1}_{2\,2}=-\frac{127825}{192\times256^2}$$

$$A^{4\,1}_{3\,1}=-\frac{119}{48\times256}$$

$$A^{4\,1}_{3\,2}=-\frac{51343}{216\times256^2}$$

$$A^{4\,1}_{4\,1}=-\frac{25}{48\times256}$$

$$A^{4\,1}_{4\,2}=-\frac{275}{8\times256^2}$$

$$A^{4\,2}_{0\,2}=\frac{741115}{5184\times256^2}$$

$$A^{4\,2}_{1\,2}=\frac{38381086}{405\times256^3}$$

$$A^{4\,2}_{2\,2}=\frac{423833}{1152\times256^2}$$

$$A^{4\,2}_{3\,2}=\frac{20845}{144\times256^2}$$

$$A^{4\,2}_{4\,2}=-\frac{173}{12\times256^2}$$

$$A^{4\,2}_{5\,2}=-\frac{30}{256^2}$$

$$A^{4\,2}_{6\,2}=-\frac{36}{5\times256^2}$$

Coefficients for f_5:

$$A_{0}^{5}{}_{0}^{0}=\frac{7}{3840}$$
$$A_{0}^{5}{}_{1}^{0}=\frac{90354659}{259200\times256^2}$$
$$A_{1}^{5}{}_{0}^{0}=\frac{7}{768}$$
$$A_{1}^{5}{}_{1}^{0}=\frac{886243}{1080\times256^2}$$
$$A_{2}^{5}{}_{0}^{0}=\frac{25}{1536}$$
$$A_{2}^{5}{}_{1}^{0}=\frac{24375}{32\times256^2}$$
$$A_{3}^{5}{}_{0}^{0}=\frac{5}{384}$$
$$A_{3}^{5}{}_{1}^{0}=\frac{8950}{27\times256^2}$$
$$A_{4}^{5}{}_{0}^{0}=\frac{25}{24\times256}$$
$$A_{4}^{5}{}_{1}^{0}=\frac{1375}{24\times256^2}$$
$$A_{0}^{5}{}_{1}^{1}=-\frac{5470394}{81\times256^3}$$
$$A_{1}^{5}{}_{1}^{1}=-\frac{88735}{96\times256^2}$$
$$A_{2}^{5}{}_{1}^{1}=-\frac{25\times15587}{288\times256^2}$$
$$A_{3}^{5}{}_{1}^{1}=-\frac{25\times997}{24\times256^2}$$
$$A_{4}^{5}{}_{1}^{1}=-\frac{105}{64\times256}$$
$$A_{5}^{5}{}_{1}^{1}=-\frac{9}{32\times256}$$

Coefficients for f_6:

$$A_{0}^{6}{}_{0}^{0}=\frac{91}{432\times256}$$
$$A_{0}^{6}{}_{1}^{0}=\frac{285522407}{4860\times256^3}$$
$$A_{1}^{6}{}_{0}^{0}=\frac{91}{72\times256}$$
$$A_{1}^{6}{}_{1}^{0}=\frac{144862403}{810\times256^3}$$
$$A_{2}^{6}{}_{0}^{0}=\frac{23}{8\times256}$$
$$A_{2}^{6}{}_{1}^{0}=\frac{2553091}{2880\times256^2}$$
$$A_{3}^{6}{}_{0}^{0}=\frac{13}{1024}$$
$$A_{3}^{6}{}_{1}^{0}=\frac{56387}{96\times256^2}$$
$$A_{4}^{6}{}_{0}^{0}=\frac{15}{8\times256}$$
$$A_{4}^{6}{}_{1}^{0}=\frac{815}{4\times256^2}$$
$$A_{5}^{6}{}_{0}^{0}=\frac{9}{20\times256}$$
$$A_{5}^{6}{}_{1}^{0}=\frac{297}{10\times256^2}$$
$$A_{0}^{6}{}_{1}^{1}=-\frac{2483629}{54\times256^3}$$
$$A_{1}^{6}{}_{1}^{1}=-\frac{4371541}{5760\times256^2}$$
$$A_{2}^{6}{}_{1}^{1}=-\frac{265901}{192\times256^2}$$
$$A_{3}^{6}{}_{1}^{1}=-\frac{403535}{288\times256^2}$$
$$A_{4}^{6}{}_{1}^{1}=-\frac{79841}{96\times256^2}$$
$$A_{5}^{6}{}_{1}^{1}=-\frac{2205}{8\times256^2}$$
$$A_{6}^{6}{}_{1}^{1}=-\frac{2401}{60\times256^2}$$

Coefficients for f_7:

$$A_{0}^{7}{}_{0}^{0}=\frac{13}{126\times256}$$
$$A_{1}^{7}{}_{0}^{0}=\frac{13}{18\times256}$$
$$A_{2}^{7}{}_{0}^{0}=\frac{287}{144\times256}$$
$$A_{3}^{7}{}_{0}^{0}=\frac{833}{288\times256}$$
$$A_{4}^{7}{}_{0}^{0}=\frac{343}{144\times256}$$
$$A_{5}^{7}{}_{0}^{0}=\frac{343}{320\times256}$$
$$A_{6}^{7}{}_{0}^{0}=\frac{2401}{45\times256^2}$$

Coefficients for f_8:

$$A_{0}^{8}{}_{0}^{0}=\frac{13\times19}{18\times256^2}$$
$$A_{1}^{8}{}_{0}^{0}=\frac{13\times19}{576\times256}$$
$$A_{2}^{8}{}_{0}^{0}=\frac{89}{64\times256}$$
$$A_{3}^{8}{}_{0}^{0}=\frac{59}{24\times256}$$
$$A_{4}^{8}{}_{0}^{0}=\frac{47}{18\times256}$$
$$A_{5}^{8}{}_{0}^{0}=\frac{19}{2880}$$
$$A_{6}^{8}{}_{0}^{0}=\frac{7}{2880}$$
$$A_{7}^{8}{}_{0}^{0}=\frac{1}{2520}$$

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